

Uniqueness in inverse acoustic and electromagnetic scattering by penetrable obstacles

Jiaqing Yang, Bo Zhang and Haiwen Zhang
LSEC and Institute of Applied Mathematics
Academy of Mathematics and Systems Science
Chinese Academy of Sciences

Beijing 100190, China

(jiaqingyang@amss.ac.cn, b.zhang@amt.ac.cn, zhanghaiwen@amss.ac.cn)

Abstract

This paper considers the inverse problem of scattering of time-harmonic acoustic and electromagnetic plane waves by bounded, inhomogeneous, penetrable obstacles in a homogeneous background medium. A new method is proposed to prove the unique determination of the penetrable, inhomogeneous obstacle from the far-field pattern for all incident plane waves at a fixed frequency. Our method is based on constructing a well-posed interior transmission problem in a *small* domain associated with the Helmholtz or modified Helmholtz equation and the Maxwell or modified Maxwell equations, where a key role is played by the fact that the domain is small, which ensures, for the previously unresolved case with the transmission coefficient $\lambda = 1$ or $\lambda_H = 1$, that the lowest transmission eigenvalue is large so that a given wave number k is not an eigenvalue of the interior transmission problem. Another ingredient in our proofs is a priori estimates of solutions to the transmission scattering problems with boundary data in L^p ($1 < p < 2$), which are established in this paper by using the integral equation method. A main feature of the new method is that it can deal with the acoustic and electromagnetic cases in a unified way and can be easily applied to deal with inverse scattering by unbounded rough interfaces.

Keywords: Uniqueness, inverse scattering problem, far-field pattern, penetrable obstacle, transmission problem, interior transmission problem.

1 Introduction

Consider the problem of scattering of a time-harmonic acoustic plane wave by an inhomogeneous penetrable obstacle surrounded in a homogeneous background medium. Let the bounded penetrable obstacle be denoted by D in \mathbb{R}^3 with a smooth boundary $\partial D \in C^2$ and an inhomogeneous refractive index $n \in L^\infty(D)$ such that $\operatorname{Re}[n(x)] > 0$ and $\operatorname{Im}[n(x)] \geq 0$. Then the scattering

problem is modeled by

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D} := D_e, \quad (1.1)$$

$$\Delta v + k^2 n v = 0 \quad \text{in } D, \quad (1.2)$$

$$u - v = 0, \quad \frac{\partial u}{\partial \nu} - \lambda \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial D, \quad (1.3)$$

$$\lim_{r \rightarrow \infty} r \left(\frac{\partial u^s}{\partial r} - i k u^s \right) = 0, \quad r = |x|, \quad (1.4)$$

where $\lambda > 0$ is the transmission coefficient depending on the properties of the media in D and D_e , $u := u^i + u^s$ denotes the total field in $D_e = \mathbb{R}^3 \setminus \overline{D}$, $u^i = e^{ikx \cdot d}$ is the incident plane wave, u^s is the scattered wave, and ν is the unit normal on ∂D directed into the exterior of D . Here, the wave number $k > 0$ is given by $k = \omega/c$ with the frequency $\omega > 0$ and the sound speed $c > 0$ and $d \in \mathbb{S}^2$ is the incident direction.

The condition (1.4) is referred to the *Sommerfeld radiation condition* which allows the following asymptotic behavior of the scattered field u^s :

$$u^s(x; d) = \frac{e^{ik|x|}}{|x|} \left\{ u^\infty(\hat{x}; d) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \rightarrow \infty \quad (1.5)$$

uniformly in all directions $\hat{x} = x/|x| \in \mathbb{S}^2$, where u^∞ is defined on the unit sphere \mathbb{S}^2 and known as the far field pattern of the scattered field u^s .

We also consider the problem of scattering of a time-harmonic electromagnetic plane wave by the inhomogeneous penetrable obstacle D surrounded in a homogeneous background medium. This problem can be formulated as follows:

$$\text{curl } E - ikH = 0, \quad \text{curl } H + ikE = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \quad (1.6)$$

$$\text{curl } G - ikF = 0, \quad \text{curl } F + ikn(x)G = 0 \quad \text{in } D, \quad (1.7)$$

$$\nu \times E - \nu \times G = 0, \quad \nu \times H - \lambda_H \nu \times F = 0 \quad \text{on } \partial D, \quad (1.8)$$

$$\lim_{r \rightarrow \infty} r(H^s \times \hat{x} - E^s) = 0, \quad r = |x|, \quad (1.9)$$

where E, G are the electric fields, H, F are the magnetic fields, $E = E^i + E^s$ and $H = H^i + H^s$ in $\mathbb{R}^3 \setminus \overline{D}$ with the incident plane wave

$$E^i(x) = \frac{i}{k} \text{curl curl } p e^{ikx \cdot d}, \quad H^i(x) = \frac{1}{ik} \text{curl } E^i(x).$$

Here, $\lambda_H = \mu_0/\mu_1$, $k^2 = \omega^2 \varepsilon_0 \mu_0$ is the wave number, $n = (\varepsilon_1 + i\sigma_1/\omega)\mu_1/(\varepsilon_0 \mu_0)$ is the refractive index in the inhomogeneous penetrable obstacle D with electric permittivity ε_1 , magnetic permeability μ_1 and electric conductivity $\sigma_1 \geq 0$ differing from the electric permittivity ε_0 , magnetic permeability μ_0 and electric conductivity $\sigma_0 = 0$ of the surrounding medium D_e , d is the incident direction and p is the polarization vector. In addition, the condition (1.9) is known as the *Silver-Müller radiation condition*, which leads to the asymptotic behaviors:

$$E^s(x) = \frac{e^{ik|x|}}{|x|} \left\{ E^\infty(\hat{x}; d; p) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \rightarrow \infty \quad (1.10)$$

$$H^s(x) = \frac{e^{ik|x|}}{|x|} \left\{ H^\infty(\hat{x}; d; p) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \rightarrow \infty \quad (1.11)$$

uniformly for all $\hat{x} = x/|x| \in \mathbb{S}^2$, where E^∞ and $H^\infty (= \hat{x} \times E^\infty)$ defined on \mathbb{S}^2 are called the far field patterns of the electric field E^s and the magnetic field H^s , respectively.

The existence of a unique solution to the transmission scattering problems (1.1)-(1.4) and (1.6)-(1.9) can be established by the variational approach or integral equation methods [25, 26, 27, 28, 29]. In this paper, we will assume that the transmission scattering problems (1.1)-(1.4) and (1.6)-(1.9) are well-posed and study the *inverse scattering problem*: given k , λ or λ_H , determine the obstacle D and the refractive index n from a knowledge of $u^\infty(\hat{x}; d)$ or $E^\infty(\hat{x}; d; p)$ for all $\hat{x}, d \in S^2$ and $p \in \mathbb{R}^3$. After the penetrable obstacle D is uniquely determined, the unique identification of the refractive index n can be established easily by using the ideas from [10, 17, 35, 36]. Thus, in this paper we only consider the inverse problem of determining the obstacle D from the far-field data without knowing n in advance.

The first uniqueness result for penetrable obstacles was established by Isakov [19] in 1990. The idea is to construct singular solutions of the scattering problem with respect to two different penetrable obstacles with identical far-field patterns, based on the variational method. In 1993, Kirsch and Kress [23] greatly simplified the method of Isakov by using the integral equation technique to establish a priori estimates of the solution on some part of the interface ∂D . In [23] the method was also extended to the case of Neumann boundary conditions (corresponding to impenetrable, sound-hard obstacles). Since then, the idea has been extensively studied and applied to establish uniqueness results for many other inverse scattering problems with transmission or conductive boundary conditions as well as other boundary conditions (see, e.g., [12, 14, 20, 24, 25, 27, 29, 31, 37] and the references quoted there).

The idea has also been extended to establish uniqueness results for inverse electromagnetic scattering problems by a penetrable, inhomogeneous, isotropic obstacle D in [16] under the condition that the boundary ∂D is in $C^{2,\alpha}$ with $0 < \alpha < 1$, the refractive index $n \in C^{1,\alpha}(\overline{D})$ is a constant near the boundary ∂D and $\text{Im}(n(x_0)) > 0$ for some $x_0 \in D$, by a penetrable, homogeneous, isotropic obstacle coated with a thin conductive layer in [14], and by a penetrable, homogeneous, isotropic obstacle D with buried obstacles in [26] under the condition that n is a complex constant with positive imaginary part in D . However, it is difficult to extend the method to the case of inhomogeneous, anisotropic media. To overcome this difficulty, in [18] Hähner introduced a different technique to prove the unique determination of a penetrable, inhomogeneous, anisotropic obstacle D from a knowledge of the scattered near-fields for all incident plane acoustic waves. The method of Hähner is based on a study of the existence, uniqueness and regularity of solutions to the corresponding interior transmission problem in D . In [2] Cakoni and Colton extended Hähner's idea to deal with the case with a penetrable, inhomogeneous, anisotropic obstacle possibly partly coated with a thin layer of a highly conductive material. It seems difficult to apply the idea in [2, 18] to the case with multi-layered obstacles. Recently in [11], Elschner and Hu considered the inverse transmission scattering problem by a two-dimensional, impenetrable obstacle surrounded by an unknown piecewise homogeneous medium and proved that the far-field patterns for all incident and observation directions at a fixed frequency uniquely determine the unknown surrounding medium as well as the impenetrable obstacle. Their method is based on constructing the Green function to a two-dimensional elliptic equation with piecewise constant leading coefficients associated with the direct scattering problem and studying the singularity of the Green function when the point source position approaches the interfaces and the impenetrable obstacle. The method in [11] also works for the three-dimensional case and the case with periodic structures. However, the method is difficult to be extended to the case of Maxwell's equations.

It should be pointed out that all the above uniqueness results were obtained under the assumption that the transmission coefficient $\lambda \neq 1$ or $\lambda_H \neq 1$ for the isotropic case or the matrix characterizing the anisotropic medium is different from the identity matrix I . In this paper, we propose a new technique to deal with the case when $\lambda = 1$ or $\lambda_H = 1$ and prove that the penetrable obstacle D can be uniquely determined from the far-field data for all incident plane waves at a fixed frequency for this case. Our method is based on the construction of a well-posed interior transmission problem in a *small* domain inside D associated with the Helmholtz or Maxwell equations. Here, a key role is played by the smallness of the domain which ensures, for the case $\lambda = 1$ or $\lambda_H = 1$, that the lowest transmission eigenvalue is large so that a given wave number k is not an eigenvalue of the constructed interior transmission problem. This is different from the method used in [2, 18], where the interior transmission problem considered is defined in the *whole penetrable obstacle* D and may have interior transmission eigenvalues, so the case $\lambda = 1$ or $\lambda_H = 1$ is excluded. For the case $\lambda \neq 1$ or $\lambda_H \neq 1$, our method gives a simplified proof. Furthermore, our method is also extended to the electromagnetic case in Section 3 and to the case of unbounded interfaces in [30].

It is well known that the existence and distribution of the eigenvalues of interior transmission problems play an important role in the linear sampling method [3] and the factorization method [22]. Thus, the existence and computation of the eigenvalues of interior transmission problems have been extensively studied recently (see, e.g. [3, 4, 5, 6, 34] and the references there). In particular, it was proved in [4] that the lowest transmission eigenvalue trends to infinity as the radius of the domain in which the interior transmission problem is defined trends to zero. Thus, for a given wave number k the domain can be taken to be small enough so that k is not an eigenvalue of the interior transmission problem. Our method is motivated by this observation.

The remaining part of the paper is organized as follows. In Sections 2 and 3 we consider the inverse acoustic and electromagnetic scattering problems by penetrable obstacles, respectively. In the appendix, we utilize the integral equation method to establish a priori estimates of solutions to the acoustic and electromagnetic transmission problems with boundary values in L^p ($1 < p < 2$), which are used in our uniqueness proofs of the inverse scattering problems. It is expected that these a priori estimates are also useful in other applications.

2 The inverse acoustic scattering problem

In this section we introduce the new technique to prove the unique determination of the inhomogeneous penetrable obstacle D from the far-field pattern $u^\infty(\hat{x}; d)$ for all $\hat{x}, d \in S^2$. Our method is based on constructing a well-posed interior transmission problem in a *small* domain associated with the Helmholtz or modified Helmholtz equation. Here, a key role is played by the smallness of the domain which ensures that the given wave number k is not a transmission eigenvalue of the constructed interior transmission problem for the case $\lambda = 1$. It should be noted that all the previous methods do not work for the case $\lambda = 1$. For the case $\lambda \neq 1$ which has been considered previously, our method gives a simplified proof. Furthermore, our method also works for the electromagnetic case, as shown in the next section, and for the case of unbounded interfaces, as seen in [30].

2.1 Interior transmission problems

Let $\Omega \subset \mathbb{R}^3$ be a simply connected and bounded domain with $\partial\Omega \in C^2$. If $\lambda = 1$ we consider the following interior transmission problem (ITP):

$$\Delta U + k^2 n(x)U = 0 \quad \text{in } \Omega, \quad (2.1)$$

$$\Delta V + k^2 V = 0 \quad \text{in } \Omega, \quad (2.2)$$

$$U - V = f_1, \quad \frac{\partial U}{\partial \nu} - \frac{\partial V}{\partial \nu} = f_2 \quad \text{on } \partial\Omega, \quad (2.3)$$

where $f_1 \in H^{1/2}(\partial\Omega)$ and $f_2 \in H^{-1/2}(\partial\Omega)$. This problem has been studied in [4].

Let $w := U - V$. Then it is easy to see that w satisfies the fourth-order equation

$$(\Delta + k^2 n) \frac{1}{n-1} (\Delta + k^2) w = 0 \quad (2.4)$$

with the boundary conditions $\gamma_0 w = f_1$ and $\gamma_1 w = f_2$. Here, γ_j ($j = 0, 1$) denotes the j th-order trace operator.

Define the Hilbert space

$$H_\Delta^1(\Omega) = \{w \in H^1(\Omega) : \Delta w \in L^2(\Omega)\}$$

with the norm $\|w\|_{H_\Delta^1(\Omega)}^2 = \|w\|_{H^1(\Omega)}^2 + \|\Delta w\|_{L^2(\Omega)}^2$. Using the Green's theorem, we easily prove that $\gamma_0 w \in H^{1/2}(\partial\Omega)$, $\gamma_1 w \in H^{-1/2}(\partial\Omega)$. In particular, if $\gamma_0 w = \gamma_1 w = 0$ for all $w \in H_\Delta^1(\Omega)$, then $H_\Delta^1(\Omega) = H_0^2(\Omega)$.

We assume that the data $f_1 \in H^{1/2}(\partial\Omega)$ and $f_2 \in H^{-1/2}(\partial\Omega)$ satisfy the condition (C): there exists a $w_0 \in H_\Delta^1(\Omega)$ such that $\gamma_0 w_0 = f_1$, $\gamma_1 w_0 = f_2$ and

$$\|w_0\|_{H_\Delta^1(\Omega)} \leq C(\|f_1\|_{H^{1/2}(\partial\Omega)} + \|f_2\|_{H^{-1/2}(\partial\Omega)}). \quad (2.5)$$

Then the interior transmission problem (ITP) is equivalent to the variational problem: Find $w \in H_\Delta^1(\Omega)$ with $\gamma_0 w = f_1$ and $\gamma_1 w = f_2$ such that

$$a(w, h) := \int_\Omega \frac{1}{n-1} (\Delta + k^2) w (\Delta + k^2 n) \bar{h} dx = 0 \quad \text{for all } h \in H_0^2(\Omega). \quad (2.6)$$

Let $\tilde{w} := w - w_0 \in H_0^2(\Omega)$. Then the variational problem (2.6) is equivalent to the problem: Find $\tilde{w} \in H_0^2(\Omega)$ such that

$$a(\tilde{w}, h) = -a(w_0, h) \quad \text{for all } h \in H_0^2(\Omega). \quad (2.7)$$

Based on (2.7), the following result can be established (see [4] for a proof).

Lemma 2.1. ([4, Lemma 2.4]) If $n(x) > 1 + r_0$ or $0 < n(x) < 1 - r_1$, then

$$a(\tilde{w}, \tilde{w}) \geq C \|\tilde{w}\|_{H_0^2(\Omega)}, \quad \forall \tilde{w} \in H_0^2(\Omega)$$

for $0 < k^2 < \min\{\lambda_1(\Omega), \lambda_1(\Omega)/\sup(n)\}$, where $r_0, r_1 > 0$ and $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of the operator $-\Delta$ in Ω .

By Lemma 2.1 the following result can be easily obtained.

Corollary 2.2. *For any fixed $k > 0$, if the diameter of Ω is small enough (so $\lambda_1(\Omega)$ is large enough) so that $k^2 < \min\{\lambda_1(\Omega), \lambda_1(\Omega)/\sup(n)\}$, then the interior transmission problem (ITP) has a unique solution $(U, V) \in L^2(\Omega) \times L^2(\Omega)$ with*

$$\|U\|_{L^2(\Omega)} + \|V\|_{L^2(\Omega)} \leq C(\|f_1\|_{H^{1/2}(\partial\Omega)} + \|f_2\|_{H^{-1/2}(\partial\Omega)}). \quad (2.8)$$

Proof. For any fixed $k > 0$, if the diameter of Ω is small enough so that $k^2 < \min\{\lambda_1(\Omega), \lambda_1(\Omega)/\sup(n)\}$, then, by Lemma 2.1 it follows that

$$a(\tilde{w}, \tilde{w}) \geq C\|\tilde{w}\|_{H_0^2(\Omega)}^2 \quad \text{for all } \tilde{w} \in H_0^2(\Omega)$$

This, together with the Lax-Milgram theorem, implies that the variational problem (2.7) has a unique solution $\tilde{w} \in H_0^2(\Omega)$ satisfying the estimate

$$\|\tilde{w}\|_{H_0^2(\Omega)} \leq C(\|f_1\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|f_2\|_{H^{-\frac{1}{2}}(\partial\Omega)}). \quad (2.9)$$

Define $U := [1/(n-1)](\Delta + k^2)w$, $V := U - w$. Then it is easy to see that $(U, V) \in L^2(\Omega) \times L^2(\Omega)$, with $U - V \in H_0^2(\Omega)$, is the unique solution to the interior transmission problem (ITP). The estimate (2.8) follows easily from (2.9) and the fact that $w = \tilde{w} + w_0$. \square

If $\lambda \neq 1$ we consider the following modified interior transmission problem (MITP):

$$\Delta U - U = \rho_1 \quad \text{in } \Omega, \quad (2.10)$$

$$\Delta V - V = \rho_2 \quad \text{in } \Omega, \quad (2.11)$$

$$U - V = f_1, \quad \lambda \frac{\partial U}{\partial \nu} - \frac{\partial V}{\partial \nu} = f_2 \quad \text{on } \partial\Omega, \quad (2.12)$$

where $\rho_0, \rho_1 \in L^2(\Omega)$, $f_1 \in H^{1/2}(\partial\Omega)$ and $f_2 \in H^{-1/2}(\partial\Omega)$. This problem has been studied in [3], and the following result was obtained (see [3, Theorem 6.7]).

Lemma 2.3. ([3, Theorem 6.7]) If $\lambda \neq 1$ then the problem (MITP) has a unique solution $(U, V) \in H^1(\Omega) \times H^1(\Omega)$ such that

$$\|U\|_{H^1(\Omega)} + \|V\|_{H^1(\Omega)} \leq C(\|\rho_1\|_{L^2(\Omega)} + \|\rho_2\|_{L^2(\Omega)} + \|f_1\|_{H^{1/2}(\partial\Omega)} + \|f_2\|_{H^{-1/2}(\partial\Omega)}).$$

2.2 Uniqueness of the inverse problem

Based on Corollary 2.2 and Lemma 2.3, we can establish the following global uniqueness result for the inverse acoustic scattering problem.

Theorem 2.4. *Given $k > 0$, let $u^\infty(\hat{x}; d)$ and $\tilde{u}^\infty(\hat{x}; d)$ be the far-field patterns of the scattering solutions to the scattering problem (1.1)–(1.4) with respect to the penetrable obstacles D with the refractive index $n \in L^\infty(D)$ and \tilde{D} with the refractive index $\tilde{n} \in L^\infty(\tilde{D})$, respectively. Assume that $u^\infty(\hat{x}; d) = \tilde{u}^\infty(\hat{x}; d)$ for all $\hat{x}, d \in \mathbb{S}^2$. Then $D = \tilde{D}$.*

Proof. Assume that $D \neq \tilde{D}$. Without loss of generality, choose $z^* \in \partial D \setminus \partial \tilde{D}$ and define

$$z_j := z^* + (\delta/j)\nu(z^*), \quad j = 1, 2, \dots$$

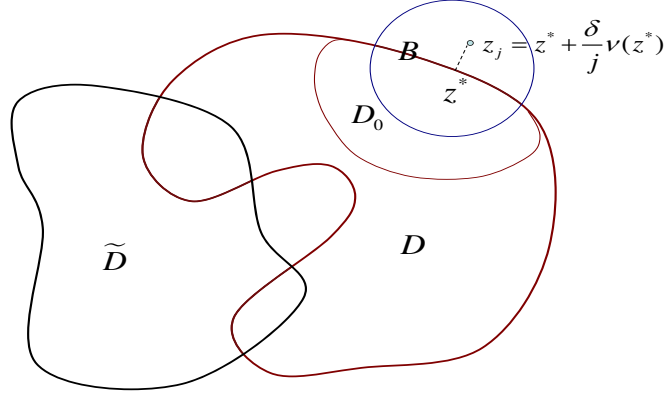


Figure 1: Two different scattering obstacles

with a sufficiently small $\delta > 0$ such that $z_j \in B$, where B denotes a small ball centered at z^* such that $B \cap \overline{\widetilde{D}} = \emptyset$. See Fig. 1.

Case 1: $\lambda \neq 1$. Let (u_j, v_j) and $(\widetilde{u}_j, \widetilde{v}_j)$ be the unique solution to the transmission scattering problem (1.1)-(1.4) with respect to D with refractive index n and \widetilde{D} with refractive index \widetilde{n} , respectively, corresponding to the incident point source

$$u_j^i(x) := \Phi(x, z_j) = \frac{\exp(ik|x - z_j|)}{4\pi|x - z_j|}, \quad j = 1, 2, \dots$$

The assumption that $u^\infty(\widehat{x}; d) = \widetilde{u}^\infty(\widehat{x}; d)$ for all $\widehat{x}, d \in \mathbb{S}^2$, together with Rellich's lemma and the denseness result [7, Theorems 5.4 and 5.5], implies that

$$u_j^s(x) = \widetilde{u}_j^s(x) \quad \text{in } \overline{G}, \quad j = 1, 2, \dots, \quad (2.13)$$

where G denotes the unbounded component of $\mathbb{R}^3 \setminus \overline{(D \cup \widetilde{D})}$.

Since $z^* \in \partial D \setminus \partial \widetilde{D}$ and $\partial D \in C^2$, there is a small smooth (C^2) domain D_0 such that $(B \cap D) \subset D_0 \subset (D \setminus \overline{\widetilde{D}})$. Define $U_j := v_j$, $V_j := \widetilde{u}_j$ in D_0 . Then (U_j, V_j) satisfies the modified interior transmission problem (MITP) with $\Omega = D_0$ and

$$\begin{aligned} \rho_1(j) &:= -(k^2 n + 1)v_j|_{D_0}, & \rho_2(j) &:= -(k^2 + 1)\widetilde{u}_j|_{D_0}, \\ f_1(j) &:= (v_j - \widetilde{u}_j)|_{\partial D_0}, & f_2(j) &:= (\lambda \partial v_j / \partial \nu - \partial \widetilde{u}_j / \partial \nu)|_{\partial D_0}. \end{aligned}$$

From (2.13) it is clear that $f_1(j) = f_2(j) = 0$ on $\Gamma_1 := \partial D_0 \cap \partial D$. Since z^* has a positive distance from \widetilde{D} , the well-posedness of the scattering problem (1.1)-(1.4) implies that

$$\|\widetilde{u}_j^s\|_{H^2(D_0)} \leq C \quad \text{uniformly for } j \in \mathbb{N}. \quad (2.14)$$

This implies that $\rho_2(j) \in L^2(D_0)$ is uniformly bounded for $j \in \mathbb{N}$ since $\Phi(\cdot, z_j) \in L^2(D_0)$ is uniformly bounded for $j \in \mathbb{N}$. From Corollary A.2 it is known that $\rho_1(j)$ is uniformly bounded in $L^2(D_0)$ for $j \in \mathbb{N}$.

We now prove that $f_1(j)$ and $f_2(j)$ are uniformly bounded in $H^{1/2}(\partial D_0)$ and $H^{-1/2}(\partial D_0)$, respectively, for $j \in \mathbb{N}$. To this end, define $w_j := v_j - \tilde{u}_j^s - \Phi(\cdot, z_j) = v_j - \tilde{u}_j$. Then $w_j \in H^2(D \setminus \overline{\tilde{D}})$ for every $j \in \mathbb{N}$ since $z_j \in \mathbb{R}^3 \setminus \overline{\tilde{D}}$ and is a solution to the problem

$$\Delta w_j = g_j \quad \text{in } D \setminus \overline{\tilde{D}}, \quad w_j|_{\Gamma_1} = 0,$$

where $g_j := k^2[\Phi(\cdot, z_j) + \tilde{u}_j^s - nv_j] \in L^2(D \setminus \overline{\tilde{D}})$. It follows from [13, Theorem 9.13] that

$$\|w_j\|_{H^2(D_0)} \leq C(\|w_j\|_{L^2(D \setminus \overline{\tilde{D}})} + \|g_j\|_{L^2(D \setminus \overline{\tilde{D}})}) \leq C \quad (2.15)$$

uniformly for $j \in \mathbb{N}$. Since $f_1(j) = w_j|_{\partial D_0}$ and

$$f_2(j) = \lambda \partial w_j / \partial \nu + (\lambda - 1)[\partial \tilde{u}_j^s / \partial \nu + \partial \Phi(\cdot, z_j) / \partial \nu],$$

it easily follows, by using (2.14), (2.15) and the fact that $f_1(j)|_{\Gamma_1} = f_2(j)|_{\Gamma_1} = 0$, that $f_1(j)$ and $f_2(j)$ are uniformly bounded in $H^{1/2}(\partial D_0)$ and $H^{-1/2}(\partial D_0)$, respectively, for $j \in \mathbb{N}$. Thus, by Lemma 2.3 we have

$$\|\Phi(\cdot, z_j)\|_{H^1(D_0)} - \|\tilde{u}_j^s\|_{H^1(D_0)} \leq \|\tilde{u}_j\|_{H^1(D_0)} = \|V_j\|_{H^1(D_0)} \leq C.$$

However, this is a contradiction since $\|\tilde{u}_j^s\|_{H^1(D_0)}$ is uniformly bounded and $\|\Phi(\cdot, z_j)\|_{H^1(D_0)} \rightarrow \infty$ as $j \rightarrow \infty$. Hence, $D = \tilde{D}$.

Case 2: $\lambda = 1$. In this case, we use the incident point source of higher-order:

$$u_j^i(x) := \nabla_x \Phi(x, z_j) \cdot \vec{a}, \quad j = 1, 2, \dots,$$

where $\vec{a} \in \mathbb{R}^3$ is a fixed vector, and let (u_j, v_j) and $(\tilde{u}_j, \tilde{v}_j)$ be the unique solution to the transmission scattering problem (1.1)-(1.4) with respect to D with refractive index n and \tilde{D} with refractive index \tilde{n} , respectively, corresponding to the incident wave $u^i(x) = u_j^i(x)$. Similarly as in Case 1, by Rellich's lemma and the denseness result [7, Theorems 5.4 and 5.5], it again follows, from the assumption $u^\infty(\hat{x}; d) = \tilde{u}^\infty(\hat{x}; d)$ for all $\hat{x}, d \in \mathbb{S}^2$, that

$$u_j^s(x) = \tilde{u}_j^s(x) \quad \text{in } \overline{G}, \quad j = 1, 2, \dots \quad (2.16)$$

Then $(U_j, V_j) := (v_j, \tilde{u}_j)$ satisfies the interior transmission problem (ITP) with $\Omega = D_0$ and

$$f_1(j) := (v_j - \tilde{u}_j)|_{\partial D_0}, \quad f_2(j) := (\partial v_j / \partial \nu - \partial \tilde{u}_j / \partial \nu)|_{\partial D_0}.$$

From (2.16) it follows that $f_1(j) = f_2(j) = 0$ in Γ_1 .

In order to utilize Corollary 2.2 to derive a contradiction, we need to verify that $f_1(j), f_2(j)$ satisfy the condition (C). To this end, we choose a cut-off function $\chi \in C_0^\infty(\mathbb{R}^3)$ such that

$$\chi(x) = \begin{cases} 0, & \text{in } \mathbb{R}^3 \setminus B, \\ 1, & \text{in } B_1, \end{cases}$$

where B_1 is a small ball centered at z^* satisfying that $B_1 \subsetneq B$. Define the function

$$w_0(j) := [1 - \chi(x)](v_j - \tilde{u}_j).$$

It is easy to see that $w_0(j) \in H_{\Delta}^1(D_0)$ for every $j \in \mathbb{N}$ with $w_0(j)|_{\partial D_0} = f_1(j)$ and $\partial w_0(j)/\partial \nu|_{\partial D_0} = f_2(j)$.

We now prove that $\|w_0(j)\|_{H_{\Delta}^1(D_0)}$ is uniformly bounded for $j \in \mathbb{N}$. Since z^* has a positive distance from \tilde{D} , we have, by the well-posedness of the scattering problem (1.1)-(1.4), that

$$\|\tilde{u}_j^s\|_{H_{\Delta}^1(D_0)} \leq C \quad (2.17)$$

uniformly for $j \in \mathbb{N}$. It further follows from Theorem A.4 that

$$\|v_j - u_j^i\|_{H^1(D_0)} \leq C \quad (2.18)$$

uniformly for $j \in \mathbb{N}$. This, combined with (2.17), yields that

$$\|w_0(j)\|_{H^1(D_0)} \leq \|v_j - \tilde{u}_j\|_{H^1(D_0)} = \|v_j - u_j^i - \tilde{u}_j^s\|_{H^1(D_0)} \leq C. \quad (2.19)$$

It remains to prove that $\Delta w_0(j)$ is uniformly bounded in $L^2(D_0)$. By a direct computation, it is found that

$$\Delta w_0(j) = \Delta(1 - \chi)(v_j - \tilde{u}_j) + 2\nabla(1 - \chi) \cdot \nabla(v_j - \tilde{u}_j) + (1 - \chi)\Delta(v_j - \tilde{u}_j).$$

In view of (2.19) the first and second terms are uniformly bounded in $L^2(D_0)$. Since $\|u_j^i\|_{L^2(D_0 \setminus \overline{B_1})} = \|\nabla \Phi(\cdot, z_j) \cdot \vec{a}\|_{L^2(D_0 \setminus \overline{B_1})} \leq C$, and by (2.17) and (2.18), we have

$$\Delta(v_j - \tilde{u}_j) = k^2[\tilde{u}_j^s - n(v_j - u_j^i)] + k^2(1 - n)u_j^i$$

is uniformly bounded in $L^2(D_0 \setminus \overline{B_1})$ for $j \in \mathbb{N}$. This, together with the fact that $\chi|_{B_1} = 0$, implies that $\|\Delta w_0(j)\|_{L^2(D_0)} \leq C$ uniformly for $j \in \mathbb{N}$. This combined with (2.19) gives the uniform boundedness of $\|w_0(j)\|_{H_{\Delta}^1(D_0)}$. Thus, $f_1(j), f_2(j)$ satisfy the condition (C).

It is well known that the smallest Dirichlet eigenvalue $\lambda_1(D_0)$ of $-\Delta$ in D_0 tends to $+\infty$ as the diameter ρ of D_0 goes to zero. Thus, D_0 can be chosen such that its diameter ρ is sufficiently small so $k^2 < \min\{\lambda_1(D_0), \lambda_1(D_0)/\sup(n)\}$. This, together with Corollary 2.2, implies that $\|V_j\|_{L^2(D_0)} = \|\tilde{u}_j\|_{L^2(D_0)}$ is uniformly bounded for $j \in \mathbb{N}$. Thus,

$$\|\nabla_x \Phi(\cdot, z_j) \cdot \vec{a}\|_{L^2(D_0)} - \|\tilde{u}_j^s\|_{L^2(D_0)} \leq \|\tilde{u}_j\|_{L^2(D_0)} \leq C \quad (2.20)$$

uniformly for $j \in \mathbb{N}$. On choosing $\vec{a} = \nu(z^*)$, it is easy to see that

$$\int_{D_0} |\nabla_x \Phi(\cdot, z_j) \cdot \nu(z^*)|^2 dx \geq \frac{C}{j^2} \int_{D_0} \frac{1}{|x - z_j|^6} = O(j).$$

This combined with (2.20) leads to a contradiction as $j \rightarrow \infty$. Thus, $D = \tilde{D}$. The proof of the theorem is then completed. \square

Remark 2.5. (i) By making use of Remark A.3, Theorem 2.4 can be easily extended to the two-dimensional case.

(ii) The method can be easily extended to the case of multi-layered media with or without buried obstacles. In particular, for the case with buried obstacles, the method can be used in conjunction with the technique in [25] to establish a uniqueness result on the simultaneous identification of the interfaces and buried obstacles.

3 The inverse electromagnetic scattering problem

In this section, we apply the technique introduced in Section 2 to obtain similar uniqueness results for the inverse electromagnetic scattering by penetrable obstacles. To this end, we introduce the interior transmission problems for the time-harmonic Maxwell equations.

3.1 Interior transmission problems

For the case $\lambda_H = \mu_0/\mu_1 = 1$, we consider the interior transmission problem (ITPM):

$$\operatorname{curl} \operatorname{curl} E_0 - k^2 E_0 = 0 \quad \text{in } \Omega, \quad (3.1)$$

$$\operatorname{curl} \operatorname{curl} F_0 - k^2 n(x) F_0 = 0 \quad \text{in } \Omega, \quad (3.2)$$

$$F_0 \times \nu - E_0 \times \nu = h_1 \quad \text{on } \partial\Omega, \quad (3.3)$$

$$\operatorname{curl} F_0 \times \nu - \operatorname{curl} E_0 \times \nu = h_2 \quad \text{on } \partial\Omega. \quad (3.4)$$

To study the well-posedness of the above problem (ITPM), we introduce the Hilbert spaces

$$H(\operatorname{curl}, \Omega) := \{\mathbf{u} \in L^2(\Omega)^3 : \operatorname{curl} \mathbf{u} \in L^2(\Omega)^3\},$$

$$H_0(\operatorname{curl}, \Omega) := \{\mathbf{u} \in H(\operatorname{curl}, \Omega) : \mathbf{u} \times \nu = 0\}$$

with the inner product $(\mathbf{u}, \mathbf{v})_{\operatorname{curl}} = (\mathbf{u}, \mathbf{v})_{L^2} + (\operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v})_{L^2}$, and the Hilbert spaces

$$\mathcal{U}(\Omega) := \{\mathbf{u} \in H(\operatorname{curl}, \Omega) : \operatorname{curl} \mathbf{u} \in H(\operatorname{curl}, \Omega)\},$$

$$\mathcal{U}_0(\Omega) := \{\mathbf{u} \in H_0(\operatorname{curl}, \Omega) : \operatorname{curl} \mathbf{u} \in H_0(\operatorname{curl}, \Omega)\}$$

with the inner product $(\mathbf{u}, \mathbf{v})_{\mathcal{U}} = (\mathbf{u}, \mathbf{v})_{\operatorname{curl}} + (\operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v})_{\operatorname{curl}}$.

We also need the following assumption (**H**): the data (h_1, h_2) satisfies that there always exists a $\mathbf{w} \in \mathcal{U}(\Omega)$ such that

$$\mathbf{w} \times \nu = h_1, \quad \operatorname{curl} \mathbf{w} \times \nu = h_2 \quad \text{on } \partial\Omega. \quad (3.5)$$

Define the set $\tau(\partial\Omega)$ which consists of (h_1, h_2) satisfying the property (3.5) and is equipped with the norm

$$\|(h_1, h_2)\|_{\tau(\partial\Omega)} := \inf\{\|\mathbf{w}\|_{\mathcal{U}(\Omega)} : \mathbf{w} \times \nu = h_1, \operatorname{curl} \mathbf{w} \times \nu = h_2 \text{ on } \partial\Omega\}.$$

Definition 3.1. *If $(E_0, F_0) \in L^2(\Omega)^3 \times L^2(\Omega)^3$ satisfies the interior transmission problem (ITPM) in the distribution sense such that $F_0 - E_0 \in \mathcal{U}(\Omega)$, then (E_0, F_0) is called a weak solution of the interior transmission problem (ITPM).*

Let $\mathbf{u} := F_0 - E_0$. Then \mathbf{u} satisfies

$$(\operatorname{curl} \operatorname{curl} - k^2 n) \frac{1}{n-1} (\operatorname{curl} \operatorname{curl} - k^2) \mathbf{u} = 0 \quad \text{in } \Omega, \quad (3.6)$$

$$\mathbf{u} \times \nu = h_1, \quad \operatorname{curl} \mathbf{u} \times \nu = h_2 \quad \text{on } \partial\Omega. \quad (3.7)$$

It is easy to see that the interior transmission problem (ITPM) is equivalent to the variational problem: Find $\mathbf{u} \in \mathcal{U}$ with the boundary condition (3.7) such that

$$b(\mathbf{u}, \mathbf{v}) = 0 \quad \text{for all } \mathbf{v} \in \mathcal{U}_0(\Omega), \quad (3.8)$$

where

$$b(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \frac{1}{n-1} (\operatorname{curl} \operatorname{curl} \mathbf{u} - k^2 \mathbf{u}) \cdot (\operatorname{curl} \operatorname{curl} \bar{\mathbf{v}} - k^2 n \bar{\mathbf{v}}) dx.$$

Let $\tilde{\mathbf{u}} := \mathbf{u} - \mathbf{w}$. Then $\tilde{\mathbf{u}} \in \mathcal{U}_0(\Omega)$ and (3.8) is equivalent to the problem

$$b(\tilde{\mathbf{u}}, \mathbf{v}) = -b(\mathbf{w}, \mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathcal{U}_0(\Omega). \quad (3.9)$$

Based on (3.9), the following result can be obtained (see [4]).

Lemma 3.2. ([4, Lemma 2.9]) *If $n > 1 + r_0$ or $0 < n < 1 - r_1$, then*

$$b(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) \geq C \|\tilde{\mathbf{u}}\|_{\mathcal{U}(\Omega)}^2$$

for $0 < k^2 < \min\{\lambda_1(\Omega), \lambda_1(\Omega)/\sup(n)\}$, where $r_0, r_1 > 0$, and $\lambda_1(\Omega)$ is the smallest Dirichlet eigenvalue of $-\Delta$ in Ω .

By Lemma 3.2 we have the following corollary.

Corollary 3.3. *For any fixed $k > 0$, if the diameter of Ω is small enough (so $\lambda_1(\Omega)$ is large enough) so that $k^2 < \min\{\lambda_1(\Omega), \lambda_1(\Omega)/\sup(n)\}$, then the interior transmission problem (ITPM) has a unique solution $(E_0, F_0) \in L^2(\Omega)^3 \times L^2(\Omega)^3$ with*

$$\|E_0\|_{L^2(\Omega)^3} + \|F_0\|_{L^2(\Omega)^3} \leq C \|(h_1, h_2)\|_{\tau(\partial\Omega)}. \quad (3.10)$$

Proof. It is clear that $-b(\mathbf{w}, \mathbf{v})$ defines a bounded, linear functional on $\mathcal{U}_0(\Omega)$. By Lemma 3.2 and the Lax-Milgram theorem, it is easy to see that there exists a unique solution $\tilde{\mathbf{u}} \in \mathcal{U}_0(\Omega)$ such that

$$\|\tilde{\mathbf{u}}\|_{\mathcal{U}_0(\Omega)} \leq C \|\mathbf{w}\|_{\mathcal{U}(\Omega)},$$

where $C > 0$ is independent of the choice of \mathbf{w} . This, combined with the fact that $\mathbf{u} = \tilde{\mathbf{u}} + \mathbf{w}$, gives

$$\|\mathbf{u}\|_{\mathcal{U}(\Omega)} \leq (C + 1) \|\mathbf{w}\|_{\mathcal{U}(\Omega)},$$

which implies that

$$\|\mathbf{u}\|_{\mathcal{U}(\Omega)} \leq (C + 1) \inf_{\mathbf{w}} \|\mathbf{w}\|_{\mathcal{U}(\Omega)} = (C + 1) \|(h_1, h_2)\|_{\tau(\partial\Omega)}.$$

Define $F_0 := [1/(n-1)](\operatorname{curl} \operatorname{curl} - k^2)\mathbf{u}$ and $E_0 := F_0 - \mathbf{u}$. Then $(E_0, F_0) \in L^2(\Omega)^3 \times L^2(\Omega)^3$ and satisfies the interior transmission problem (ITPM) with the estimate (3.10). \square

In the case $\lambda_H = \mu_0/\mu_1 \neq 1$, we consider the modified interior transmission problem (MITPM):

$$\operatorname{curl} \operatorname{curl} E_0 + E_0 = \xi_1 \quad \text{in } \Omega, \quad (3.11)$$

$$\operatorname{curl} \operatorname{curl} F_0 + F_0 = \xi_2 \quad \text{in } \Omega, \quad (3.12)$$

$$(F_0)_T - (E_0)_T = h_1 \times \nu \quad \text{on } \partial\Omega, \quad (3.13)$$

$$\nu \times \operatorname{curl} F_0 - \lambda_H \nu \times \operatorname{curl} E_0 = h_2 \quad \text{on } \partial\Omega, \quad (3.14)$$

where $(\cdot)_T = \nu \times (\cdot) \times \nu$, $\xi_1, \xi_2 \in L^2(\Omega)^3$ and $h_1, h_2 \in Y(\partial\Omega)$. Here,

$$Y(\partial\Omega) = \{\mathbf{u} \in H^{-1/2}(\partial\Omega) : \nu \cdot \mathbf{u} = 0, \operatorname{Div}(\mathbf{u}) \in H^{-1/2}(\partial\Omega)\}$$

denotes the trace space of $H(\operatorname{curl}, \Omega)$.

Lemma 3.4. ([2, Theorem 3.3]) *If $\lambda_H \neq 1$, then the modified interior transmission problem (MITPM) admits a unique solution $(E_0, F_0) \in H(\text{curl}, \Omega) \times H(\text{curl}, \Omega)$ satisfying the estimate*

$$\|E_0\|_{H(\text{curl}, \Omega)} + \|F_0\|_{H(\text{curl}, \Omega)} \leq C(\|\xi_1\|_{L^2(\Omega)^3} + \|\xi_2\|_{L^2(\Omega)^3} + \|h_1\|_{Y(\partial\Omega)} + \|h_2\|_{Y(\partial\Omega)}).$$

3.2 Uniqueness of the inverse problem

We now make use of Corollary 3.3 and Lemma 3.4 to establish the following global uniqueness result for the inverse electromagnetic scattering problem.

Theorem 3.5. *Given $k > 0$, let $E^\infty(\hat{x}; d; p)$ and $\tilde{E}^\infty(\hat{x}; d; p)$ be the electric far-field patterns with respect to the scattering problem (1.6) – (1.9) with the penetrable obstacles D with the refractive index $n \in C^1(\overline{D})$ and \tilde{D} with the refractive index $\tilde{n} \in C^1(\overline{\tilde{D}})$, respectively. If $E^\infty(\hat{x}; d; p) = \tilde{E}^\infty(\hat{x}; d; p)$ for all $\hat{x}, d \in \mathbb{S}^2$ and all polarizations $p \in \mathbb{R}^3$, then $D = \tilde{D}$.*

Proof. Assume that $D \neq \tilde{D}$. Without loss of generality, choose $z^* \in \partial D \setminus \partial \tilde{D}$ and define

$$z_j := z^* + (\delta/j)\nu(z^*), \quad j = 1, 2, \dots$$

with a sufficiently small $\delta > 0$ such that $z_j \in B$, where B denotes a small ball centered at z^* and satisfies that $B \cap \overline{\tilde{D}} = \emptyset$. See Fig. 1.

It is easy to see that the scattering problem (1.6)-(1.9) can be reformulated in terms of the electric fields E and G as:

$$\text{curl curl } E - k^2 E = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \quad (3.15)$$

$$\text{curl curl } G - k^2 n(x)G = 0 \quad \text{in } D, \quad (3.16)$$

$$\nu \times E = \nu \times G \quad \text{on } \partial D, \quad (3.17)$$

$$\nu \times \text{curl } E = \lambda_H \nu \times \text{curl } G \quad \text{on } \partial D, \quad (3.18)$$

with $E = E^i + E^s$ in $\mathbb{R}^3 \setminus \overline{D}$ and the Silver-Müller radiation condition

$$\text{curl } E^s(x) \times \hat{x} - ikE^s(x) = O\left(\frac{1}{|x|^2}\right), \quad |x| \rightarrow \infty. \quad (3.19)$$

Case 1: $\lambda_H = \mu_0/\mu_1 \neq 1$. We consider the incident magnetic dipole wave in the form

$$E_j^i(x) = \frac{\text{curl}(p\Phi(x, z_j))}{\|\text{curl}(p\Phi(x, z_j))\|_{L^2(\partial D)}} \quad j = 1, 2, 3, \dots$$

with polarization $p \in \mathbb{R}^3$. Let (E_j, G_j) and $(\tilde{E}_j, \tilde{G}_j)$ be the unique solution of the scattering problem (3.15)-(3.19) with respect to D with refractive index n and \tilde{D} with refractive index \tilde{n} , respectively, corresponding to the incident magnetic dipole $E_j^i(x)$. From the assumption $E^\infty(\hat{x}; d; p) = \tilde{E}^\infty(\hat{x}; d; p)$ for all $\hat{x}, d \in \mathbb{S}^2$ and all polarizations $p \in \mathbb{R}^3$, and by using Rellich's lemma and the denseness results, it follows that

$$E_j^s(x) = \tilde{E}_j^s(x) \quad \text{in } \overline{G_0}, \quad (3.20)$$

where G_0 denotes the unbounded component of $\mathbb{R}^3 \setminus (\overline{D \cup \tilde{D}})$.

Since $z^* \in \partial D \setminus \widetilde{D}$ and $\partial D \in C^2$, we can choose a small C^2 -smooth domain D_0 such that $(B \cap D) \subset D_0 \subset (D \setminus \widetilde{D})$. Let $E_0(j) = G_j$ and $F_0(j) = \widetilde{E}_j$ in D_0 . Then $E_0(j)$ and $F_0(j)$ satisfy the modified interior transmission problem (MITPM) with $\Omega = D_0$ and

$$\begin{aligned}\xi_1(j) &:= (k^2 n + 1)G_j, \\ \xi_2(j) &:= (k^2 + 1)\widetilde{E}_j, \\ h_1(j) &:= \nu \times \widetilde{E}_j - \nu \times G_j, \\ h_2(j) &:= \nu \times \operatorname{curl} \widetilde{E}_j - \frac{\mu_0}{\mu_1} \nu \times \operatorname{curl} G_j.\end{aligned}$$

From (3.20) and the transmission conditions on ∂D , it is easily seen that $h_1(j) = h_2(j) = 0$ on $\Gamma_1 := \partial D_0 \cap \partial D$. Further, from the well-posedness of the scattering problem (1.6)-(1.9), and in view of the positive distance from z^* to \widetilde{D} , we know that

$$\|\widetilde{E}_j^s\|_{H(\operatorname{curl}, D_0)} \leq C, \quad (3.21)$$

where C is independent of $j \in \mathbb{N}$. Noting that $\|E_j^i\|_{L^2(D_0)}$ is uniformly bounded for all $j \in \mathbb{N}$, and by (3.21), we deduce that $\xi_2(j)$ is bounded in $L^2(D_0)$ uniformly for all $j \in \mathbb{N}$. Moreover, by Theorem B.1 it follows that

$$\|G_j\|_{L^2(D_0)^3} + \|G_j\|_{H(\operatorname{curl}, D_0 \setminus \overline{B_{z^*}})} \leq C,$$

uniformly for all $j \in \mathbb{N}$, where B_{z^*} is chosen such that $(B_{z^*} \cap D) \subsetneq D_0$. Thus, $\xi_1(j)$ is bounded in $L^2(D_0)$ uniformly for $j \in \mathbb{N}$, and by (3.21) $\widetilde{E}_j - G_j$ and $\widetilde{E}_j - \lambda_H G_j$ are bounded in $H(\operatorname{curl}, D_0 \setminus \overline{B_{z^*}})$ uniformly for $j \in \mathbb{N}$. Then, by the trace theorem and the fact that $h_1(j) = h_2(j) = 0$ on Γ_1 , we have

$$\|h_1(j)\|_{Y(\partial D_0)} + \|h_2(j)\|_{Y(\partial D_0)} \leq C$$

uniformly for $j \in \mathbb{N}$. Then, by Lemma 3.4 it is derived that

$$\|E_j^i\|_{H(\operatorname{curl}, D_0)} - \|\widetilde{E}_j^s\|_{H(\operatorname{curl}, D_0)} \leq \|\widetilde{E}_j\|_{H(\operatorname{curl}, D_0)} = \|F_0(j)\|_{H(\operatorname{curl}, D_0)} \leq C.$$

Choosing $p = \nu(z^*)$ we have

$$\begin{aligned}\|\operatorname{curl} E_j^i\|_{L^2(D_0)} &= \frac{\|\operatorname{curl} \operatorname{curl} [\nu(z^*)\Phi(x, z_j)]\|_{L^2(D_0)}}{\|\operatorname{curl} [\nu(z^*)\Phi(x, z_j)]\|_{L^2(\partial D_0)}} \\ &\geq \frac{\|\nabla \operatorname{div} [\nu(z^*)\Phi(x, z_j)]\|_{L^2(D_0)}}{\|\operatorname{curl} [\nu(z^*)\Phi(x, z_j)]\|_{L^2(\partial D_0)}} - \frac{\|k^2 [\nu(z^*)\Phi(x, z_j)]\|_{L^2(D_0)}}{\|\operatorname{curl} [\nu(z^*)\Phi(x, z_j)]\|_{L^2(\partial D_0)}} := I + II.\end{aligned}$$

Obviously, the second term II is uniformly bounded due to the boundedness of $\Phi(x, z_j)$ in $L^2(D_0)$. Without loss of generality, we assume that $z^* = (0, 0, 0)$ and $\nu(z^*) = (0, 0, 1)$ (in fact, other cases can be easily transformed into this one by a linear transformation). Then we have

$$I^2 = \int_{D_0} \frac{x_1^2 + x_2^2}{|x - z_j|^8} dx \Big/ \int_{\partial D_0} \frac{x_1^2 + x_2^2}{|x - z_j|^6} ds(x) = O(j^3), \quad j \rightarrow \infty.$$

This implies that $\|E_j^i\|_{H(\operatorname{curl}, D_0)} \rightarrow \infty$ as $j \rightarrow \infty$. However, this is a contradiction since $\|\widetilde{E}_j^s\|_{H(\operatorname{curl}, D_0)}$ is uniformly bounded for all $j \in \mathbb{N}$. Therefore, we have $D = \widetilde{D}$.

Case 2: $\lambda_H = \mu_0/\mu_1 = 1$. In this case, we consider the incident magnetic dipole wave

$$E_j^i(x) = \operatorname{curl}(p\Phi(x, z_j)) \quad j = 1, 2, 3, \dots,$$

with $p \in \mathbb{R}^3$. Let (E_j, G_j) and $(\widetilde{E}_j, \widetilde{G}_j)$ denote the unique solution to the scattering problem (3.15)-(3.19) with respect to D with refractive index n and \widetilde{D} with refractive index \widetilde{n} , respectively, corresponding to the incident magnetic dipole $E_j^i(x)$. Similarly as in Case 1, we have, from the assumption $E^\infty(\widehat{x}; d; p) = \widetilde{E}^\infty(\widehat{x}; d; p)$ for all $\widehat{x}, d \in \mathbb{S}^2$ and all polarizations $p \in \mathbb{R}^3$, that

$$E_j^s(x) = \widetilde{E}_j^s(x) \quad \text{in } \overline{G_0}. \quad (3.22)$$

Define $E_0(j) := \widetilde{E}_j$ and $F_0(j) := G_j$ in D_0 . Then $(E_0(j), F_0(j))$ satisfies the interior transmission problem (ITPM) with $\Omega = D_0$ and

$$\begin{aligned} h_1(j) &:= \nu \times G_j - \nu \times \widetilde{E}_j \\ h_2(j) &:= \nu \times \operatorname{curl} G_j - \nu \times \operatorname{curl} \widetilde{E}_j. \end{aligned}$$

From (3.22) we see that $h_1(j) = h_2(j) = 0$ in $\Gamma_1 := \partial D_0 \cap \partial D$. Choose the cut-off function $\chi \in C_0^\infty(\mathbb{R}^3)$ such that

$$\chi(x) = \begin{cases} 0, & x \in \mathbb{R}^3 \setminus \overline{B}, \\ 1, & x \in B_1. \end{cases}$$

Here, $B_1 \subsetneq B$ is a small ball centered at z^* . Define $W(j) := [1 - \chi](G_j - \widetilde{E}_j)$. Then it is easy to check that $\nu \times W(j)|_{\partial D_0} = h_1(j)$ and $\nu \times \operatorname{curl} W(j)|_{\partial D_0} = h_2(j)$.

Since z^* has a positive distance from \widetilde{D} , we then have by the well-posedness of the scattering problem (1.6)-(1.9) (or (3.15)-(3.19)) that

$$\|\widetilde{E}_j^s\|_{\mathcal{U}(D_0)} \leq C. \quad (3.23)$$

On the other hand, by Theorem B.2 it follows that

$$\|G_j - E_j^i\|_{H(\operatorname{curl}, D_0)} + \|G\|_{L^p(D)} \leq C \|\operatorname{curl}(p\Phi(x, z))\|_{L^p(D)} \leq C_1 \quad (3.24)$$

for $6/5 \leq p < 3/2$, where C, C_1 are independent of $j \in \mathbb{N}$. This, combined with (3.23), yields that

$$\|W(j)\|_{H(\operatorname{curl}, D_0)} \leq C. \quad (3.25)$$

It remains to prove that $\operatorname{curl} \operatorname{curl} W(j)$ is uniformly bounded in $L^2(D)^3$ for $j \in \mathbb{N}$. By a direct calculation, we find that

$$\begin{aligned} \operatorname{curl} \operatorname{curl} W(j) &= \nabla a(x) \times \operatorname{curl}(G_j - \widetilde{E}_j) + a(x) \operatorname{curl} \operatorname{curl}(G_j - \widetilde{E}_j) \\ &\quad + \operatorname{curl}[\nabla a(x) \times (G_j - \widetilde{E}_j)] \end{aligned}$$

with $a(x) := 1 - \chi(x)$. From (3.23) and (3.24) it is seen that the first term $\nabla a \times \operatorname{curl}(G_j - \widetilde{E}_j) \in L^2(D_0)$. From the Maxwell equations, it is found that

$$a \operatorname{curl} \operatorname{curl}(G_j - \widetilde{E}_j) = k^2 a [n(G_j - E_j^i) - \widetilde{E}_j^s + (n - 1)E_j^i] \in L^2(D_0) \quad (3.26)$$

since $a|_{B_1} = 0$ and $\|aE_j^i\|_{L^2(D_0)^3} \leq C$. Further, we have

$$\begin{aligned} \operatorname{curl} [\nabla a(x) \times (G_j - \widetilde{E}_j)] &= \nabla a \operatorname{div} (G_j - \widetilde{E}_j) - (\nabla a \cdot \nabla)(G_j - \widetilde{E}_j) \\ &\quad + [(G_j - \widetilde{E}_j) \cdot \nabla] \nabla a - (G_j - \widetilde{E}_j) \Delta a. \end{aligned}$$

(3.23) and (3.24) imply that the third and forth terms on the right-hand side of the above equation are uniformly bounded in $L^2(D_0)$ for $j \in \mathbb{N}$. For the first and second terms, since $\nabla a|_{B_1} = 0$, we only need to show that $\nabla(G_j - \widetilde{E}_j)$ and $\operatorname{div}(G_j - \widetilde{E}_j)$ are uniformly bounded in $L^2(D_0 \setminus \overline{B_1})$. First, from (B.16) in Appendix B it is noted that

$$G_j - E_j^i = T_2 G_j - T_1(G_j - E_j^i) - T_1 E_j^i,$$

where T_1, T_2 are defined in Appendix B (just after (B.16)). Obviously, $T_2 G_j \in W^{2,p}(D) \hookrightarrow H^1(D)$ and $T_1 G_j^s \in H^1(D)$ since, by (3.24), $G_j \in L^p(D)$ with $6/5 \leq p < 3/2$ and $(G_j - E_j^i) \in L^2(D)$. For $x \in D_0 \setminus \overline{B_1}$, it is easy to see that $\|T_1 E_j^i\|_{H^1(D \setminus \overline{B_1})} \leq C$. Then we have

$$\|G_j - E_j^i\|_{H^1(D \setminus \overline{B_1})} \leq C.$$

This, together with the estimate $\|\widetilde{E}_j^s\|_{H^1(D_0)} \leq C$, implies that $\nabla(G_j - \widetilde{E}_j)$ and $\operatorname{div}(G_j - \widetilde{E}_j)$ are uniformly bounded in $L^2(D_0 \setminus \overline{B_1})$, and then

$$\operatorname{curl} [\nabla a(x) \times (G_j - \widetilde{E}_j)] \in L^2(D_0)$$

uniformly for $j \in \mathbb{N}$. Therefore, we have that $\|\operatorname{curl} \operatorname{curl} W(j)\|_{L^2(D_0)} \leq C$, so

$$\|(h_1(j), h_2(j))\|_{\tau(\partial D_0)} \leq \|W(j)\|_{\mathcal{U}(D_0)} \leq C, \quad (3.27)$$

where C is independent of $j \in \mathbb{N}$.

Similarly as in the acoustic case, D_0 can be chosen so that $k^2 < \min\{\lambda_1(D_0), \lambda_1(D_0)/\sup(n)\}$. It then follows from (3.27) and Lemma 3.2 that

$$\|\widetilde{E}_j\|_{L^2(D_0)} = \|E_0(j)\|_{L^2(D_0)} \leq C_1 \|(h_1(j), h_2(j))\|_{\tau(\partial D_0)} \leq C$$

uniformly for $j \in \mathbb{N}$. Therefore, we have

$$\|\operatorname{curl}(p\Phi(\cdot, z_j))\|_{L^2(D_0)} - \|\widetilde{E}_j^s\|_{L^2(D_0)} \leq \|\widetilde{E}_j\|_{L^2(D_0)} \leq C$$

uniformly for $j \in \mathbb{N}$. This is a contradiction since $\|\operatorname{curl}(p\Phi(\cdot, z_j))\|_{L^2(D_0)} \rightarrow \infty$ as $j \rightarrow \infty$ and $\|\widetilde{E}_j^s\|_{L^2(D_0)}$ is bounded uniformly for $j \in \mathbb{N}$. Then we have $D = \widetilde{D}$. The proof is thus completed. \square

Remark 3.6. (i) The method can be applied to improve the uniqueness result in [17] by relaxing the smoothness requirement on the boundary ∂D and the refractive index n ($\partial D \in C^2$ instead of $\partial D \in C^{2,\alpha}$ and $n \in C^1(\overline{D})$ instead of $n \in C^{1,\alpha}(\overline{D})$, where $0 < \alpha < 1$) and by removing the assumption that the refractive index n is a constant near the boundary ∂D and $\operatorname{Im}(n(x_0)) > 0$ for some $x_0 \in D$.

(ii) The method can be easily extended to the case of multi-layered media with or without buried obstacles. In particular, for the case with buried obstacles, the method can be used in conjunction with the technique in [26] to establish a uniqueness result on the simultaneous identification of the interface and the buried obstacle without the assumption that the refractive index n is a constant with $\operatorname{Im}(n) > 0$ in D .

Appendix. A priori estimates for the transmission problems with boundary values in L^p

In this appendix, we establish a priori estimates for the acoustic and electromagnetic transmission problems with boundary values in L^p ($1 < p < 2$), employing the integral equation method. These a priori estimates play an important part in the uniqueness proofs of the inverse transmission problems in the previous sections. These a priori estimates are also interesting on their own right.

A The acoustic transmission problem

We first establish the a priori estimates of solutions of the acoustic transmission problem

$$\Delta w_1 + k^2 w_1 = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \quad (\text{A.1})$$

$$\Delta w_2 + k^2 n(x) w_2 = 0 \quad \text{in } D, \quad (\text{A.2})$$

$$w_1 - \gamma w_2 = f_1, \quad \frac{\partial w_1}{\partial \nu} - \frac{\partial w_2}{\partial \nu} = f_2 \quad \text{on } \partial D, \quad (\text{A.3})$$

$$\frac{\partial w_1}{\partial r} - ikw_1 = o\left(\frac{1}{r}\right) \quad r = |x| \rightarrow \infty, \quad (\text{A.4})$$

where $f_1, f_2 \in L^p(\partial D)$ with $1 < p < 2$ and $\gamma = 1/\lambda$.

We introduce the single- and double-layer boundary operators

$$(S\phi)(x) := \int_{\partial D} \Phi(x, y) \phi(y) ds(y), \quad x \in \partial D,$$

$$(K\phi)(x) := \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(y)} \phi(y) ds(y), \quad x \in \partial D$$

and the their normal derivative operators

$$(K'\phi)(x) := \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(x)} \phi(y) ds(y), \quad x \in \partial D,$$

$$(T\phi)(x) := \frac{\partial}{\partial \nu(x)} \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(y)} \phi(y) ds(y), \quad x \in \partial D.$$

It follows from [32, Lemma 9] and [33, Lemma 1] that the operators S, K, K' and T are bounded in $L^q(\partial D)$ ($1 < q < \infty$), and moreover, the operators S, K, K' are also compact in $L^q(\partial D)$ ($1 < q < \infty$).

Theorem A.1. *For $f_1, f_2 \in L^p(\partial D)$ with $4/3 \leq p \leq 2$ the transmission problem (A.1) – (A.4) has a unique solution $(w_1, w_2) \in L^2_{loc}(\mathbb{R}^3 \setminus \overline{D}) \times L^2(D)$ satisfying that*

$$\|w_1\|_{L^2_{loc}(\mathbb{R}^3 \setminus \overline{D})} + \|w_2\|_{L^2(D)} \leq C(\|f_1\|_{L^p(\partial D)} + \|f_2\|_{L^p(\partial D)}). \quad (\text{A.5})$$

Proof. Step 1. Assume that $k^2 n(x) \equiv k_1^2 > 0$ is a constant. Then we seek a solution of the problem (A.1)-(A.4) in the form

$$w_1(x) = \int_{\partial D} \Phi(x, y) \phi(y) ds(y) + \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(y)} \psi(y) ds(y), \quad x \in \mathbb{R}^3 \setminus \overline{D} \quad (\text{A.6})$$

$$w_2(x) = \int_{\partial D} \Phi_1(x, y) \phi(y) ds(y) + \int_{\partial D} \frac{\partial \Phi_1(x, y)}{\partial \nu(y)} \psi(y) ds(y), \quad x \in D, \quad (\text{A.7})$$

where $\Phi(x, y) = \exp(ik|x - y|)/(4\pi|x - y|)$ and $\Phi_1(x, y) = \exp(ik_1|x - y|)/(4\pi|x - y|)$.

Then, by the jump relations of the layer potentials (see [33] for the case in L^p and [7, 8] for the case in spaces of continuous functions), the transmission problem (A.1)-(A.4) can be reduced to the system of integral equations

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix} + L \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} hf_1 \\ -f_2 \end{pmatrix} \quad \text{in } L^p(\partial D) \times L^p(\partial D), \quad (\text{A.8})$$

where $h := 2/(1 + \gamma)$ and the operator L is given by

$$L := \begin{pmatrix} h(K - \gamma K_1) & h(S - \gamma S_1) \\ T_1 - T & K'_1 - K' \end{pmatrix}.$$

Here, the operators S_1, K_1, K'_1 and T_1 are defined similarly as S, K, K' and T with the kernel $\Phi(x, y)$ replaced by $\Phi_1(x, y)$. It is easy to see that (A.8) is of Fredholm type since the operators $S, S_1, K, K_1, K'_1 - K'$ and $T_1 - T$ are compact in $L^p(\partial D)$. This, together with the uniqueness of the scattering problem (1.1)-(1.4), implies that (A.8) has a unique solution $(\psi, \phi)^T \in L^p(\partial D) \times L^p(\partial D)$ satisfying the estimate

$$\|\psi\|_{L^p(\partial D)} + \|\phi\|_{L^p(\partial D)} \leq C(\|f_1\|_{L^p(\partial D)} + \|f_2\|_{L^p(\partial D)}). \quad (\text{A.9})$$

Therefore, we obtain that

$$\begin{aligned} \left\| \int_{\partial D} \Phi_1(\cdot, y) \phi(y) ds(y) \right\|_{L^2(D)} &= \sup_{g \in L^2, \|g\|_{L^2(D)}=1} \left| \int_D \left\{ \int_{\partial D} \Phi_1(x, y) \phi(y) ds(y) \right\} g(x) dx \right| \\ &= \sup_{g \in L^2, \|g\|_{L^2(D)}=1} \left| \int_{\partial D} \left\{ \int_D \Phi_1(x, y) g(x) dx \right\} \phi(y) ds(y) \right| \\ &\leq |\partial D|^{\frac{1}{q}} \sup_{g \in L^2, \|g\|_{L^2(D)}=1} \sup_{y \in \partial D} \|\Phi_1(\cdot, y)\|_{L^2(D)} \|g\|_{L^2(D)} \|\phi\|_{L^p(\partial D)} \\ &= |\partial D|^{\frac{1}{q}} \sup_{y \in \partial D} \|\Phi_1(\cdot, y)\|_{L^2(D)} \|\phi\|_{L^p(\partial D)} \end{aligned} \quad (\text{A.10})$$

and

$$\begin{aligned} \left\| \int_{\partial D} \frac{\partial \Phi_1(\cdot, y)}{\partial \nu(y)} \psi(y) ds(y) \right\|_{L^2(D)} &= \sup_{g \in L^2, \|g\|_{L^2(D)}=1} \left| \int_D \left\{ \int_{\partial D} \frac{\partial \Phi_1(x, y)}{\partial \nu(y)} \psi(y) ds(y) \right\} g(x) dx \right| \\ &= \sup_{g \in L^2, \|g\|_{L^2(D)}=1} \left| \int_{\partial D} \left\{ \frac{\partial}{\partial \nu(y)} \int_D \Phi_1(x, y) g(x) dx \right\} \psi(y) ds(y) \right| \\ &\leq \sup_{g \in L^2, \|g\|_{L^2(D)}=1} \left\| \frac{\partial}{\partial \nu(y)} \int_D \Phi_1(x, \cdot) g(x) dx \right\|_{L^q(\partial D)} \cdot \|\psi\|_{L^p(\partial D)} \\ &\leq \sup_{g \in L^2, \|g\|_{L^2(D)}=1} C \|g\|_{L^2(D)} \cdot \|\psi\|_{L^p(\partial D)} = C \|\psi\|_{L^p(\partial D)} \end{aligned} \quad (\text{A.11})$$

with $1/p + 1/q = 1$. Here, we have used the fact that the volume potential operator is bounded from $L^2(D)$ into $W^{2,2}(D)$ (see [13, Theorem 9.9]), and the boundary trace operator is bounded from $W^{1,2}(D)$ into $L^q(\partial D)$ for $2 \leq q \leq 4$ (see [1, Theorem 5.36]). Then the desired estimate (A.5) follows from (A.6)-(A.7) and (A.9)-(A.11) in the case when $k^2 n(x) \equiv k_1^2$.

Step 2. For the general case $n \in L^\infty(D)$, we consider the following problem

$$\Delta W_1 + k^2 W_1 = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \quad (\text{A.12})$$

$$\Delta W_2 + k^2 n(x) W_2 = g \quad \text{in } D, \quad (\text{A.13})$$

$$W_1 - \gamma W_2 = 0, \quad \frac{\partial W_1}{\partial \nu} - \frac{\partial W_2}{\partial \nu} = 0 \quad \text{on } \partial D, \quad (\text{A.14})$$

$$\frac{\partial W_1}{\partial r} - ik W_1 = o\left(\frac{1}{r}\right) \quad r = |x| \rightarrow \infty, \quad (\text{A.15})$$

where $g := (k_1^2 - k^2 n(x)) \tilde{w}_2 \in L^2(D)$ and $(\tilde{w}_1, \tilde{w}_2)$ denotes the solution of the problem (A.1)-(A.4) with $k^2 n(x) \equiv k_1^2$. By Step 1, we have

$$\|\tilde{w}_1\|_{L^2_{loc}(\mathbb{R}^3 \setminus \overline{D})} + \|\tilde{w}_2\|_{L^2(D)} \leq C(\|f_1\|_{L^p(\partial D)} + \|f_2\|_{L^p(\partial D)}). \quad (\text{A.16})$$

By using the variational method, it can be easily proved that for every $g \in L^2(D)$ the problem (A.12)-(A.15) has a unique solution $(W_1, W_2) \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{D}) \times H^1(D)$ satisfying the estimate

$$\|W_1\|_{H^1_{loc}(\mathbb{R}^3 \setminus \overline{D})} + \|W_2\|_{H^1(D)} \leq C\|g\|_{L^2(D)}. \quad (\text{A.17})$$

Define $w_1 := W_1 + \tilde{w}_1$ and $w_2 := W_2 + \tilde{w}_2$. Then from (A.16) and (A.17) it follows that $(w_1, w_2) \in L^2_{loc}(\mathbb{R}^3 \setminus \overline{D}) \times L^2(D)$ is the unique solution of the problem (A.1)-(A.4) satisfying the estimate (A.5). The proof is thus complete. \square

Corollary A.2. For $z^* \in \partial D$ and for a sufficiently small $\delta > 0$ define $z_j := z^* + (\delta/j)\nu(z^*) \in \mathbb{R}^3 \setminus \overline{D}$, $j \in \mathbb{N}$. Let $\lambda \neq 1$ and let (u_j, v_j) be the solution of the transmission problem (1.1) – (1.4) corresponding to the incident point source $u_j^i = \Phi(x, z_j)$, $j \in \mathbb{N}$. Then

$$\|v_j\|_{L^2(D)} \leq C \quad (\text{A.18})$$

uniformly for $j \in \mathbb{N}$.

Proof. Let $w_{1j} := u_j^s - \Phi(x, y_j)$ and $w_{2j} := \lambda v_j$ with $y_j := z^* - (\delta/j)\nu(z^*) \in D$. Then (w_1, w_2) satisfies the problem (A.1)-(A.4) with

$$\begin{aligned} f_1 = f_{1j} &:= -\Phi(z, z_j) - \Phi(x, y_j), \\ f_2 = f_{2j} &:= -\frac{\partial \Phi(z, z_j)}{\partial \nu(z)} - \frac{\partial \Phi(z, y_j)}{\partial \nu(z)}. \end{aligned}$$

Obviously, $f_{1j} \in L^p(\partial D)$ is uniformly bounded for $j \in \mathbb{N}$, where $1 < p < 2$. Further, from [9, Lemma 4.2] it is seen that $f_{2j} \in C(\partial D)$ is uniformly bounded for $j \in \mathbb{N}$, so $f_{2j} \in L^p(\partial D)$ is uniformly bounded for $j \in \mathbb{N}$, where $1 < p < 2$. The estimate (A.18) then follows from Theorem A.1. \square

Remark A.3. In the two-dimensional case, Corollary A.2 remains true with the estimate (A.18) replaced by the following one:

$$\|v_j\|_{H^1(D)} \leq C \quad (\text{A.19})$$

uniformly for $j \in \mathbb{N}$. In fact, in this case, the fundamental solution of the Helmholtz equation $\Delta u + k^2 u = 0$ is given by

$$\Phi(x, y) := (i/4)H_0^{(1)}(k|x - y|), \quad x \neq y,$$

where $H_0^{(1)}$ is a Hankel function of the first kind of order zero and has the logarithmic singularity. This implies that f_{1j} and f_{2j} defined in the proof of Corollary A.2 are uniformly bounded in $L^2(\partial D)$. Then, by (A.9) the solution $(\psi, \phi)^T$ of (A.8) satisfies the estimate

$$\|\psi\|_{L^2(\partial D)} + \|\phi\|_{L^2(\partial D)} \leq C(\|f_{1j}\|_{L^2(\partial D)} + \|f_{2j}\|_{L^2(\partial D)}) \leq C \quad (\text{A.20})$$

uniformly for $j \in \mathbb{N}$. On the other hand, w_2 given by (A.7) satisfies the boundary value problem

$$\begin{aligned} \Delta w + k^2 w &= 0 & \text{in } D, \\ \frac{\partial w}{\partial \nu} + iw &= f & \text{on } \partial D, \end{aligned}$$

where $f := \partial w_2 / \partial \nu|_{\partial D} + iw_2|_{\partial D} \in L^2(\partial D)$. It is easy to prove that the above boundary value problem has a unique solution $w \in H^1(D)$ such that

$$\|w\|_{H^1(D)} \leq C\|f\|_{L^2(\partial D)} \leq C[\|\psi\|_{L^2(\partial D)} + \|\phi\|_{L^2(\partial D)}] \leq C$$

uniformly for $j \in \mathbb{N}$, where we have used the boundedness in $L^2(\partial D)$ of the boundary integral operators K'_1, T_1 (see [32, Lemma 9] and [33, Lemma 1]). This, together with (A.20) and the argument in Step 2 of the proof of Corollary A.2, implies the required estimate (A.19).

Theorem A.4. *Let $\lambda = 1$ and let (u, v) be the solution of the transmission problem (1.1) – (1.4) corresponding to the supper singular point source $u^i = \nabla_x \Phi(x, z) \cdot \vec{a}$, where $z \in \mathbb{R}^3 \setminus \overline{D}$ and $\vec{a} \in \mathbb{R}^3$ is a fixed vector. Then $v \in L^p(D)$ and $v - u^i \in H^1(D)$ such that*

$$\|v\|_{L^p(D)} + \|v - u^i\|_{H^1(D)} \leq C\|\nabla_x \Phi(\cdot, z) \cdot \vec{a}\|_{L^p(D)} \quad (\text{A.21})$$

for every p with $6/5 \leq p < 2$.

Proof. Since u^i satisfies the Helmholtz equation $\Delta u^i + k^2 u^i = 0$ in D , it follows from the Green's theorem that the transmission problem (1.1)-(1.4) is equivalent to the Lippmann-Schwinger equation

$$v(x) = u^i(x) - k^2 \int_D \Phi(x, y)[1 - n(y)]v(y)dy, \quad x \in D.$$

Define the operator $T : L^p(D) \rightarrow L^p(D)$ by

$$(T\varphi)(x) := \int_D \Phi(x, y)[1 - n(y)]\varphi(y)dy, \quad x \in D.$$

Then we have

$$(I + T)v = u^i \quad \text{in } L^p(D) \quad \text{for } 1 < p < 2. \quad (\text{A.22})$$

It follows from [13, Theorem 9.9] that T is bounded from $L^p(D)$ into $W^{2,p}(D)$ and therefore is compact in $L^p(D)$. Thus, and by the uniqueness of the transmission problem (1.1)-(1.4), the operator $I + T$ is of Fredholm type with index zero. The Fredholm alternative then implies the existence of a unique solution v in $L^p(D)$ to (A.22) with the estimate

$$\|v\|_{L^p(D)} \leq C\|u^i\|_{L^p(D)} \quad \text{for } 1 < p < 2. \quad (\text{A.23})$$

From this and the embedding result that $W^{2,p}(D) \hookrightarrow H^1(D)$ for $6/5 \leq p < 2$, the required estimate (A.21) follows. \square

B The electromagnetic transmission problem

We now establish a priori estimates for the electromagnetic transmission problem (1.6)-(1.9) with $L^p(D)$ data ($1 < p \leq 2$), which is used in the proof of Theorem 3.5.

Introduce the magnetic dipole operator M and the electric dipole operator N by

$$\begin{aligned}(Ma)(x) &= \int_{\partial D} \nu(x) \times \operatorname{curl}_x \{a(y)\Phi(x, y)\} ds(y), \quad x \in \partial D, \\ (Na)(x) &= \nu(x) \times \operatorname{curl} \operatorname{curl} \int_{\partial D} \nu(y) \times b(y)\Phi(x, y) ds(y), \quad x \in \partial D.\end{aligned}$$

Similarly, we also introduce the operators M_1 and N_1 which are defined as M and N with the kernel $\Phi(x, y)$ replaced by $\Phi_1(x, y)$.

Theorem B.1. *Assume that $\lambda_H = \mu_0/\mu_1 \neq 1$. For $z^* \in \partial D$ let B_{z^*} be a small ball centered at z^* . Let $z \in B_{z^*} \cap (\mathbb{R}^3 \setminus \overline{D})$ and let (E, G) be the solution of the transmission problem (1.6) – (1.9) corresponding to the incident magnetic dipole $E^i(x) = \operatorname{curl}(p\Phi(x, z))/\|\operatorname{curl}(p\Phi(x, z))\|_{L^2(\partial D)}$. Then $G \in L^2(D)^3 \cap H(\operatorname{curl}, D \setminus \overline{B_{z^*}})$ with*

$$\|G\|_{L^2(D)^3} + \|G\|_{H(\operatorname{curl}, D \setminus \overline{B_{z^*}})} \leq C, \quad (\text{B.1})$$

where $C > 0$ is a constant and independent of z .

Proof. To prove (B.1), define $G_1(x) := G(x) - (\mu_1/\mu_0)E_1^i(x)$ for $x \in D$ with $E_1^i(x) := \operatorname{curl}(p\Phi_1(x, z))/\|\operatorname{curl}(p\Phi_1(x, z))\|_{L^2(\partial D)}$ with $k_1^2 \neq k^2$. Then $E^s|_{\mathbb{R}^3 \setminus \overline{D}}$ and $G_1|_D$ satisfy the transmission problem (TP1):

$$\operatorname{curl} \operatorname{curl} E^s - k^2 E^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \quad (\text{B.2})$$

$$\operatorname{curl} \operatorname{curl} G_1 - k^2 n(x) G_1 = g \quad \text{in } D, \quad (\text{B.3})$$

$$\nu \times E^s - \nu \times G_1 = f_1 \quad \text{on } \partial D, \quad (\text{B.4})$$

$$\nu \times \operatorname{curl} E^s - \frac{\mu_0}{\mu_1} \nu \times \operatorname{curl} G_1 = f_2 \quad \text{on } \partial D, \quad (\text{B.5})$$

with the radiation condition (1.6), where $f_1 := (\mu_1/\mu_0)\nu \times E_1^i|_{\partial D} - \nu \times E^i|_{\partial D}$, $f_2 := \nu \times \operatorname{curl} E_1^i|_{\partial D} - \nu \times \operatorname{curl} E^i|_{\partial D}$ and $g = (\mu_1/\mu_0)(k^2 n - k_1^2)E_1^i$.

Step 1. We first consider the case $k^2 n(x) \equiv k_1^2$. In this case, $g = 0$, and we seek the solution of the problem (TP1) in the form

$$E^s(x) = \mu_0 \operatorname{curl} \int_{\partial D} a(y)\Phi(x, y) ds(y) + \operatorname{curl} \operatorname{curl} \int_{\partial D} b(y)\Phi(x, y) ds(y), \quad x \in \mathbb{R}^3 \setminus \overline{D},$$

$$G_1(x) = \mu_1 \operatorname{curl} \int_{\partial D} a(y)\Phi_1(x, y) ds(y) + \operatorname{curl} \operatorname{curl} \int_{\partial D} b(y)\Phi_1(x, y) ds(y), \quad x \in D$$

with tangential fields a and b . Then the problem (TP1) is equivalent to the following system of integral equations:

$$\frac{1}{2}(\mu_0 + \mu_1)a + (\mu_0 M - \mu_1 M_1)a + (N - N_1)Pb = f_1, \quad (\text{B.6})$$

$$\frac{1}{2}k^2(\mu_0 + \mu_1)b + \mu_0\mu_1(N - N_1)Pa + k^2(\mu_1 M - \mu_0 M_1)b = f_2, \quad (\text{B.7})$$

with $Pc := \nu \times c$. It is easy to see that $f_1 \in T^2(\partial D)$ and $f_2 \in T^2(\partial D)$. Further, from the identity $\operatorname{div}_{\partial D}(\nu \times \operatorname{curl} E^i) = -\nu \cdot \operatorname{curl}^2 E^i$ it follows that $f_2 \in T_d^2(\partial D)$. Here,

$$\begin{aligned} T^2(\partial D) &:= \{\mathbf{u} \in L^2(\partial D)^2 : \nu \cdot \mathbf{u} = 0\}, \\ T_d^2(\partial D) &:= \{\mathbf{u} \in L^2(\partial D)^2 : \nu \cdot \mathbf{u} = 0, \operatorname{div}_{\partial D} \mathbf{u} \in L^2(\partial D)\}. \end{aligned}$$

Since $\partial D \in C^2$, it follows from [21] or [32] that the operators M, M_1 and $(N - N_1)P$ are compact in $T^2(\partial D)$ and $T_d^2(\partial D)$. Thus, the system (B.6)-(B.7) is of Fredholm type in the space $T^2(\partial D) \times T_d^2(\partial D)$. This, together with the uniqueness of the transmission problem (TP1), implies that the system (B.6)-(B.7) has a unique solution $(a, b) \in T^2(\partial D) \times T_d^2(\partial D)$ with the estimate

$$\|a\|_{T^2(\partial D)} + \|b\|_{T_d^2(\partial D)} \leq C(\|f_1\|_{T^2(\partial D)} + \|f_2\|_{T_d^2(\partial D)}). \quad (\text{B.8})$$

We now split G_1 into two parts $G_1^{(1)}$ and $G_1^{(2)}$, given by

$$\begin{aligned} G_1^{(1)}(x) &= \operatorname{curl} \int_{\partial D} a(y) \Phi_1(x, y) ds(y), \quad x \in D, \\ G_1^{(2)}(x) &= \operatorname{curl} \operatorname{curl} \int_{\partial D} b(y) \Phi_1(x, y) ds(y), \quad x \in D. \end{aligned}$$

Then $G_1 = \mu_1 G_1^{(1)} + G_1^{(2)}$. Moreover, by the properties of M_1 and N_1 , we have

$$\begin{aligned} \operatorname{curl} \operatorname{curl} G_1^{(1)} - k_1^2 G_1^{(1)} &= 0 \quad \text{in } D, \quad \nu \times G_1^{(1)} \in T^2(\partial D), \\ \operatorname{curl} \operatorname{curl} G_1^{(2)} - k_1^2 G_1^{(2)} &= 0 \quad \text{in } D, \quad \nu \times G_1^{(2)} \in T_d^2(\partial D). \end{aligned}$$

It is easy to see that $G_1^{(1)} \in L^2(D)$, $G_1^{(2)} \in H(\operatorname{curl}, D)$ and

$$\|G_1^{(1)}\|_{L^2(D)} + \|G_1^{(2)}\|_{H(\operatorname{curl}, D)} \leq C(\|a\|_{T^2(\partial D)} + \|b\|_{T_d^2(\partial D)}). \quad (\text{B.9})$$

Note that $f_1 \in T_d^2(\partial D \setminus \overline{B_{z^*}})$. This, together with (B.6) and the fact that M, M_1 and $(N - N_1)P$ are bounded from $T^2(\partial D)$ and $T_d^2(\partial D)$ into $T_d^2(\partial D)$, gives that $a \in T_d^2(\partial D \setminus \overline{B_{z^*}}) \cap T^2(\partial D)$. Choose a ball B_1 centered at z^* with $B_1 \subsetneq B_{z^*}$ and a cut-off function $\chi \in C_0^\infty(\mathbb{R}^3)$ supported in B_1 with $\chi = 1$ in a small neighborhood of z^* . Then $G_1^{(1)}$ can be written in the form

$$G_1^{(1)}(x) = \operatorname{curl} \int_{\partial D} \Phi_1(x, y) [1 - \chi(y)] a(y) ds + \operatorname{curl} \int_{\partial D} \Phi_1(x, y) \chi(y) a(y) ds.$$

Obviously, the first term belongs to $H(\operatorname{curl}, D)$ since $(1 - \chi(y))a(y) \in T_d^2(\partial D)$, and the second term belongs to $H(\operatorname{curl}, D \setminus \overline{B_1})$ since $B_1 \subsetneq B_{z^*}$. Combining (B.9) and (B.8) gives the required estimate (B.1) in the case $k^2 n(x) \equiv k_1^2$.

Step 2. For the general case $n \in L^\infty(D)$, we consider the transmission problem (TP2):

$$\operatorname{curl} \operatorname{curl} W - k^2 W = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \quad (\text{B.10})$$

$$\operatorname{curl} \operatorname{curl} W_1 - k^2 n(x) W_1 = g_1 \quad \text{in } D, \quad (\text{B.11})$$

$$\nu \times W - \nu \times W_1 = 0 \quad \text{on } \partial D, \quad (\text{B.12})$$

$$\nu \times \operatorname{curl} W - \frac{\mu_0}{\mu_1} \nu \times \operatorname{curl} W_1 = 0 \quad \text{on } \partial D \quad (\text{B.13})$$

with W satisfying the radiation condition (1.9), where $g_1 = g + (k^2 n - k_1^2) \tilde{G}_1 \in L^2(D)^3$ and $(\tilde{E}^s, \tilde{G}_1)$ is a solution of the transmission problem (IT1) with $k^2 n(x) \equiv k_1^2$. By Step 1 we have

$$\|\tilde{G}_1\|_{L^2(D)^3} + \|\tilde{G}_1\|_{H(\text{curl}, D \setminus \overline{B_{z^*}})} \leq C, \quad (\text{B.14})$$

where $C > 0$ is a constant and independent of z . By the variational method, it is easy to show that the problem (TP2) admits a unique solution $(W, W_1) \in H_{loc}(\text{curl}, D) \times H(\text{curl}, D)$ with

$$\|W_1\|_{H(\text{curl}, D)} \leq C \|g_1\|_{L^2(D)^3} \leq C (\|g\|_{L^2(D)^3} + \|\tilde{G}_1\|_{L^2(D)^3}). \quad (\text{B.15})$$

Define $E^s|_{\mathbb{R}^3 \setminus \overline{D}} := \tilde{E}^s + W$ and $G_1|_D := \tilde{G}_1 + W_1$. Then it is easy to check that (E^s, G_1) is the unique solution of the problem (TP2). The required estimate (B.1) thus follows from (B.14) and (B.15). This completes the proof of the theorem. \square

Theorem B.2. *Assume that $n \in C^1(\overline{D})$ and $\lambda_H = \mu_0/\mu_1 = 1$. For $z^* \in \partial D$ let B_{z^*} be a small ball centered at z^* . Let $z \in B_{z^*} \cap (\mathbb{R}^3 \setminus \overline{D})$ and Let (E, G) be the solution of the transmission problem (1.6)–(1.9) corresponding to the incident magnetic dipole $E^i(x) = \text{curl}(p\Phi(x, z))$. Then $G \in L^p(D)$, $G^s := G - E^i \in H(\text{curl}, D)$ and*

$$\|G\|_{L^p(D)} + \|G^s\|_{H(\text{curl}, D)} \leq C \|\text{curl}(p\Phi(x, z))\|_{L^p(D)}$$

for every $6/5 \leq p < 3/2$, where $C > 0$ is independent of z .

Proof. Since $\text{curl}(p\Phi(x, z))$ satisfies the Maxwell equation $\text{curl} \text{curl} E^i - k^2 E^i = 0$ in D , it follows from Green's theorem that the transmission problem (1.6)–(1.9) is equivalent to the integral equation

$$\begin{aligned} G(x) &= E^i(x) - k^2 \int_D \Phi(x, y) m(y) G(y) dy \\ &\quad + \text{grad} \int_D \frac{1}{n(y)} \text{grad} n(y) \cdot G(y) \Phi(x, y) dy, \quad x \in D, \end{aligned} \quad (\text{B.16})$$

where $m(y) := 1 - n(y)$. On the space $L^p(D)$, define the operators T_1 and T_2 by

$$\begin{aligned} (T_1 \varphi)(x) &:= k^2 \int_D \Phi(x, y) m(y) \varphi(y) dy, \quad x \in D, \\ (T_2 \varphi)(x) &:= \text{grad} \int_D \frac{1}{n(y)} \text{grad} n(y) \cdot \varphi(y) \Phi(x, y) dy, \quad x \in D. \end{aligned}$$

Then the integral equation (B.16) can be rewritten in the form

$$(I + T_1 - T_2)G = E^i \quad \text{in } D.$$

It follows from [13, Theorem 9.9] that T_1 is bounded from $L^p(D)$ into $W^{2,p}(D)$ and T_2 is bounded from $L^p(D)$ into $W^{1,p}(D)$. Therefore, T_1 and T_2 are compact operators in $L^p(D)$. This, together with the uniqueness of the transmission problem (1.6)–(1.9), implies that the operator $I + T_1 - T_2$ is of Fredholm type with index zero. The Fredholm alternative gives that the integral equation (B.16) has a unique solution $G \in L^p(D)$ with

$$\|G\|_{L^p(D)} \leq C \|E^i\|_{L^p(D)}. \quad (\text{B.17})$$

From (B.16) it is easily seen that

$$G^s = (T_2 - T_1)G \in W^{1,p}(D) \hookrightarrow L^2(D)$$

since $6/5 \leq p < 3/2$. Further, by the identity $\text{curl grad} = 0$, we have

$$\text{curl } G^s = -\text{curl}(T_1 G) \in W^{1,p}(D) \hookrightarrow L^2(D).$$

Thus we deduce that

$$\|G^s\|_{H(\text{curl}, D)} \leq C\|G\|_{L^p(D)} \leq C\|E^i\|_{L^p(D)}. \quad (\text{B.18})$$

The proof is then completed by combining (B.17) and (B.18). \square

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References

- [1] A. Adams and J.F. Fournier, *Sobolev Spaces* (2nd Edition), Elsevier, Singapore, 2003.
- [2] F. Cakoni and D. Colton, A uniqueness theorem for an inverse electromagnetic scattering problem in inhomogeneous anisotropic media, *Proc. Edinb. Math. Soc.* **46** (2003), 293-314.
- [3] F. Cakoni and D. Colton, *Qualitative Methods in Inverse Scattering Theory*, Springer, Berlin, 2006.
- [4] F. Cakoni, D. Gintides and H. Haddar, The existence of an infinite discrete set of transmission eigenvalues, *SIAM J. Math. Anal.* **42** (2010), 237-255.
- [5] F. Cakoni and H. Haddar, A variational approach for the solution of the electromagnetic interior transmission problem for anisotropic media, *Inverse Problems Imaging* **3** (2009), 389-403.
- [6] D. Colton, P. Monk and J. Sun, Analytical and computational methods for transmission eigenvalues, *Inverse Problems* **26** (2010) 045011.
- [7] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory* (2nd Edition), Springer, Berlin, 1998.
- [8] D. Colton and R. Kress, *Integral Equation Methods in Scattering Theory*, Wiley, New York, 1983.
- [9] D. Colton, R. Kress and P. Monk, Inverse scattering from an orthotropic medium, *J. Comput. Appl. Math.* **81** (1997), 269-298.
- [10] D. Colton and L. Päiväranta, The uniqueness of a solution to an inverse scattering problem for electromagnetic waves, *Arch. Rational Mech. Anal.* **119** (1992), 59-70.

- [11] J. Elschner and G. Hu, Uniqueness in inverse transmission scattering problems for multi-layered obstacles, *Inverse Problems Imaging* **5** (2011), 793-813.
- [12] T. Gerlach and R. Kress, Uniqueness in inverse obstacle scattering with conductive boundary condition, *Inverse Problems* **12** (1996), 619-625.
- [13] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order* (2nd Edition), Springer, New York, 1983.
- [14] F. Hettlich, On the uniqueness of the inverse conductive scattering problem for the Helmholtz equation, *Inverse Problems* **10** (1994), 129-144.
- [15] F. Hettlich, Uniqueness of the inverse conductive scattering problem for time-harmonic electromagnetic waves, *SIAM J. Appl. Math.* **56** (1996), 588-601.
- [16] P. Hähner, A uniqueness theorem for a transmission problem in inverse electromagnetic scattering, *Inverse Problems* **9** (1993), 667-678.
- [17] P. Hähner, On Acoustic, Electromagnetic and Elastic Scattering Problems in Inhomogeneous Media, Habilitationsschrift, Göttingen University, 1998.
- [18] P. Hähner, On the uniqueness of the shape of a penetrable anisotropic obstacle, *J. Comput. Appl. Math.* **116** (2000), 167-180.
- [19] V. Isakov, On uniqueness in the inverse transmission scattering problem, *Commun. Partial Differ. Equat.* **15** (1990), 1565-1587.
- [20] V. Isakov, On uniqueness in the general inverse transmission problem, *Commun. Math. Phys.* **280** (2008), 843-858.
- [21] A. Kirsch, Surface gradients and continuity properties for some integral operators in classical scattering theory, *Math. Methods Appl. Sci.* **11** (1989), 789-804.
- [22] A. Kirsch and N. Grinberg, *The Factorization Methods for Inverse Problems*, Oxford Univ. Press, Oxford, 2008.
- [23] A. Kirsch and R. Kress, Uniqueness in inverse obstacle scattering, *Inverse Problems* **9** (1993), pp. 285-299.
- [24] A. Kirsch and L. Päivärinta, On recovering obstacles inside inhomogeneities, *Math. Methods Appl. Sci.* **21** (1998), 619-651.
- [25] X. Liu and B. Zhang, Direct and inverse obstacle scattering problems in a piecewise homogeneous medium, *SIAM J. Appl. Math.* **70** (2010), 3105-3120.
- [26] X. Liu, B. Zhang and J. Yang, The inverse electromagnetic scattering problem in a piecewise homogeneous medium, *Inverse Problems* **26** (2010) 125001 (19pp).
- [27] X. Liu, B. Zhang and G. Hu, Uniqueness in the inverse scattering problem in a piecewise homogeneous medium, *Inverse Problems* **26** (2010) 015002 (14pp).

- [28] X. Liu and B. Zhang, A uniqueness result for the inverse electromagnetic scattering problem in a piecewise homogeneous medium, *Appl. Anal.* **88** (2009), 1339-1355.
- [29] X. Liu and B. Zhang, Inverse scattering by an inhomogeneous penetrable obstacle in a piecewise homogeneous medium *Acta Mathematica Scientia* **B32** (2012), 1281-1297.
- [30] Y. Lu and B. Zhang, Direct and inverse scattering problem by an unbounded rough interface and buried obstacles, Submitted, 2013.
- [31] D. Mitrea and M. Mitrea, Uniqueness for inverse conductivity and transmission problems in the class of Lipschitz domains, *Commun. Partial Differ. Equat.* **23** (1998), 1419-1448.
- [32] R. Potthast, A point-source method for inverse acoustic and electromagnetic obstacle scattering problems, *IMA J. Appl. Math.* **61** (1998), 119-140.
- [33] R. Potthast, On the convergence of a new Newton-type method in inverse scattering, *Inverse Problems* **17** (2001), 1419-1434.
- [34] J. Sun, Iterative methods for transmission eigenvalues, *SIAM J. Numer. Anal.* **49** (2011), 1860-1874.
- [35] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, *Ann. Math.* **125** (1987), 153-169.
- [36] G. Uhlmann, Electrical impedance tomography and Calderón's problem, *Inverse Problems* **25** (2009) 123011.
- [37] N. Valdivia, Uniqueness in inverse obstacle scattering with conductive boundary conditions, *Appl. Anal.* **83** (2004), 825-851.